LECTURE-03

The general form of a first order DE is given by

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

or, $$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

wher $M(x,y)$ and $N(x,y)$ are functions of two variables x and y.

Homogeneous DE

The DE $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be a homogeneous DE if $M(x,y)$ and $N(x,y)$ are homogeneous functions of same degree.

Example:-

(i) $y\frac{dy}{dx} = x$

(ii) $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

(iii) $x^2y\,dx + (x^3+y^3)\,dy = 0$

HOW TO SOLVE A HOMOGENEOUS DE

Let $M(x,y)\,dx + N(x,y)\,dy = 0$ be a homogeneous DE.

STEP:-1

we have,

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} \quad \text{---------- (1)}$$

STEP-II

Put $y = vx$

Diff. both sides w.r.to x,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

STEP-III

Putting these values in equation (1)

$$v + x\frac{dv}{dx} = -\frac{M(x, vx)}{N(x, vx)}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{M(x, vx)}{N(x, vx)} - v \quad \text{——— (2)}$$

STEP-IV

Equation (2) is a first order DE in v and x.

Solve it by variable and separable

Find out the value of v.

STEP-V

Finally put $v = \frac{y}{x}$.

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8. $xy' = x + y$

$$\Rightarrow x\frac{dy}{dx} = x + y$$

which is a homogeneous DE.

Now, $$\frac{dy}{dx} = \frac{x+y}{x} \quad \text{——— (1)}$$

Put $y = vx$

Diff. both sides w.r. to x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Putting these values in equation (1)

$$v + x\frac{dv}{dx} = \frac{x + vx}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = 1 + v$$

$$\Rightarrow x\frac{dv}{dx} = 1 + v - v$$

$$\Rightarrow x\frac{dv}{dx} = 1$$

$$\Rightarrow dv = \frac{dx}{x}$$

Integrating both sides.

$$\int dv = \int \frac{dx}{x}$$

$$\Rightarrow v = \ln|x| + C$$

$$\Rightarrow \frac{y}{x} = \ln|x| + C$$

$$\Rightarrow y = x[\ln|x| + C]$$

9. $y' = \frac{x^2 + y^2}{xy}$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 + y^2}{xy} \quad \text{---(1)}$$

which is a homogeneous DE.

Put $y = vx$

Diff. both sides w.r.to x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Putting in equation (1)

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{xvx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1 + v^2}{v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2-v^2}{v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1}{v}$$

$$\Rightarrow v\,dv = \frac{dx}{x}$$

Integrating both sides,

$$\int v\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \frac{v^2}{2} = \ln|x| + C$$

$$\Rightarrow \frac{y^2}{2x^2} = \ln|x| + C .$$

EXACT DIFFERENTIAL EQUATION

The DE $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be an exact DE if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Then its solution is given by

$$\boxed{\int M(x,y)\,dx + \int N(x,y)\,dy = C .}$$

(y-CONST.) (Term not cont. x in N)

NOTE:- (i) The meaning of the first integral is we have to integrate $M(x,y)$ w.r.to x by taking y variable as a constant.

(ii) The meaning of the second integral is we have to integrate those term of $N(x,y)$ which is free from x. If there is no such term then the 2nd integral does not exist

Example-

Verify that the given DE are exact or not

(i) $(x^2+y^2x)\,dx + x^2y\,dy = 0$

(ii) $x\sin y\,dx - y\sin x\,dy = 0$

Ans to (i)

$(x^2+y^2x)\,dx + x^2y\,dy = 0$

Here $M(x,y) = x^2+y^2x$ $\qquad N(x,y) = x^2y$

$\frac{\partial M}{\partial y} = 2yx$ $\qquad \frac{\partial N}{\partial x} = 2xy$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So, the given DE is an exact DE.

Ans to (ii)

$x\sin y\,dx - y\sin x\,dy = 0$

Here $M(x,y) = x\sin y$ $\qquad N(x,y) = -y\sin x$

$\frac{\partial M}{\partial y} = x\cos y$ $\qquad \frac{\partial N}{\partial x} = -y\cos x$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So, the given DE is a non exact DE.

Example:-

For what value of a and b the given DE is exact.

$(axy^2 + y^4)\,dx + (2x^2y + bxy^3) = 0$

Ans:- Here,

$M(x,y) = axy^2+y^4$ $\qquad N(x,y) = 2x^2y + bxy^3$

$\frac{\partial M}{\partial y} = 2axy + 4y^3$ $\quad$ $\frac{\partial N}{\partial x} = 4xy + by^3$

For exact-ness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow 2axy + 4y^3 = 4xy + by^3$$

Equating both sides,

$$2a = 4 \qquad \boxed{4 = b}$$

$$\Rightarrow \boxed{a = 2}$$

Thus if $a = 2$ and $b = 4$ then the given DE is exact.

NOTE:-

Given $U(x, y)$ the exact differential form is

$$\boxed{du = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy}$$

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1-TO-6

Given $U(x, y)$ find the exact DE $du = 0$

1. $U = x^2 - y^2$

Diff. U partially w.r.to x & y

$$\frac{\partial U}{\partial x} = 2x \qquad \frac{\partial U}{\partial y} = -2y$$

We have, the exact Differential form is

$$du = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

$$\Rightarrow du = 2x\,dx - 2y\,dy$$

Since, $du = 0$, $2x\,dx - 2y\,dy = 0$ is the required exact DE.

6. $u = \sin x \cosh y$

Diff. u partially w.r.to x & y

$$\frac{\partial u}{\partial x} = \cos x \cosh y \qquad \frac{\partial u}{\partial y} = \sin x \sinh y$$

We have the exact differential form is

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\Rightarrow du = \cos x \cosh y \, dx + \sin x \sinh y \, dy$$

Since. $du = 0$

So, $\cos x \cosh y \, dx + \sin x \sinh y \, dy = 0$

is the required exact DE.